Temperature Controllers

Introduction

This handout examines the performance of several systems for controlling the temperature of an oven by adjusting the heater power – a much harder task than it might first appear. The problem is useful to study in detail because its behaviour is easy to visualise and is typical of that encountered in almost every electronic system employing feedback. The final, so-called PID, design is an effective, and widely used, general-purpose controller.

Problem Description

Power $\dot{Q}$ is supplied to the oven by an electrical element, heat capacity C_h, at temperature T_h. This is linked through a thermal resistance R_{ho} to the oven, heat capacity C_o, which loses heat to the environment, temperature T_e, through the thermal resistance R_o of the enclosure. The controller monitors T_o and adjusts $\dot{Q}$ with the goal of maintaining the oven temperature T_o at a set-point T_s.

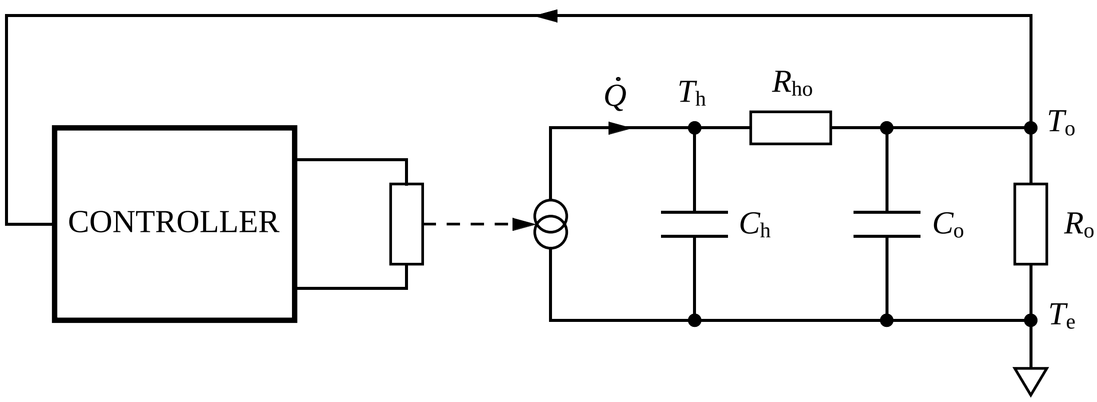

Figure 1. Thermal model of oven and controller.

The illustrative figures in this document have been drawn using parameters for a domestic cooker, estimated as follows: The heating element dissipates $\dot{Q}_{max} = 4\,\text{kW}$ when full-on. Maintaining the oven at a steady $T_o = 225\,^\circ\text{C}$ requires $\dot{Q} = 1.3\,\text{kW}$ so $R_o = 0.15\,\text{K W}^{-1}$. It takes half an hour for the oven to cool from 225 °C to 100 °C with room temperature $T_e = 25\,^\circ\text{C}$ so $C_o R_o \sim 1800\,\text{s}$ and therefore $C_o = 1.0 \times 10^4\,\text{J K}^{-1}$. The heating elements glow dull-red in operation suggesting that their temperature $T_h \sim 750\,^\circ\text{C}$ so use $R_{ho} \sim 0.1\,\text{K W}^{-1}$ and they cool with a time constant of about a minute making $C_h \sim 500\,\text{J K}^{-1}$. The on-off contol thermostat typically has a 10 °C hysteresis band.

On–Off Control

This is the simplest form of controller and is used in most domestic ovens – when $T_o < (T_s - \Delta/2)$ the heater is turned on at full power, when $T_o > (T_s + \Delta/2)$ the heater is switched off completely. The parameter Δ introduces hysteresis into the system preventing noise from switching the heater rapidly and unnecessarily when $T_o \sim T_s$. As can be seen from figure 2, due to the heat capacity of the electrical element, the fluctuations in temperature are significantly wider than Δ, the deliberately introduced hysteresis band.

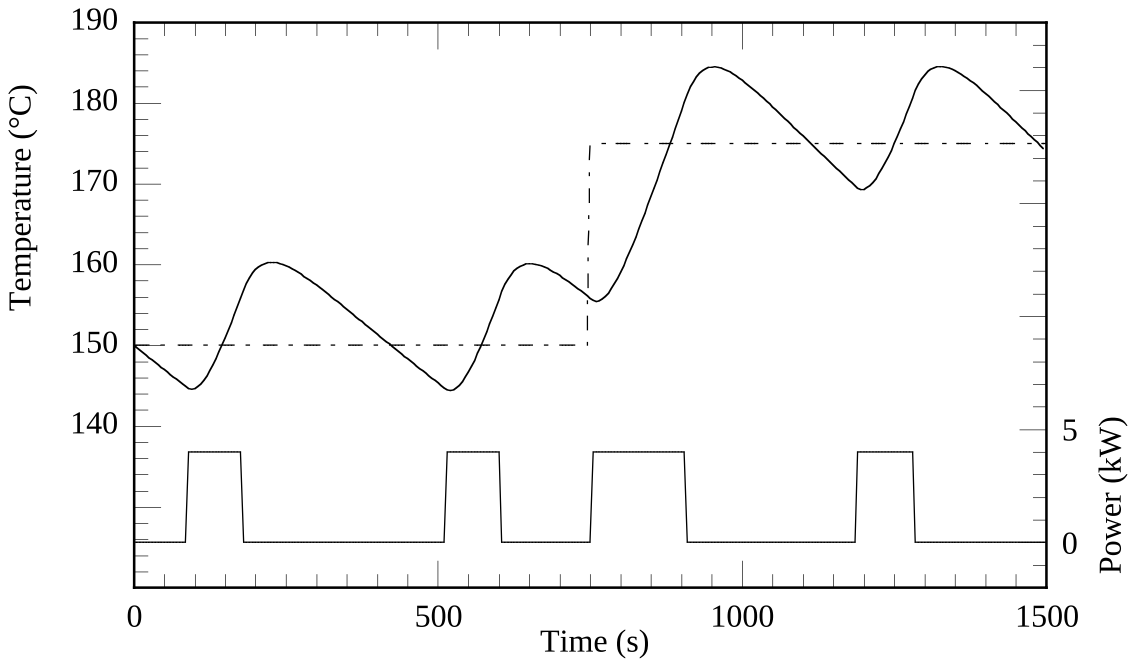

Figure 2. The oven (T_o, solid) and set-point (T_s, dashed) temperatures using an on-off controller with hysteresis band $\Delta = 10\,^\circ$C.

Operating Equations

The temperatures T_o and T_h are functions only of their initial values, $\dot{Q}$ and time. Inspection of figure 1, the equivalent thermal "circuit", shows that they are governed by two coupled differential equations

$$\dot{T}_h = \frac{\dot{Q}}{C_h} - \frac{(T_h - T_o)}{C_h R_{ho}} \qquad \text{and} \qquad \dot{T}_o = \frac{(T_h - T_o)}{C_o R_{ho}} + \frac{(T_e - T_o)}{C_o R_o}. \tag{1,2}$$

Differentiating equation 2 and eliminating terms involving T_h gives

$$[C_h R_o C_o R_{ho}]\ddot{T}_o + [C_o R_o + C_h R_o + C_h R_{ho}]\dot{T}_o + T_o = \dot{Q} R_o + T_e \tag{3}$$

which is simply the equation of motion of a forced damped harmonic oscillator. It is helpful to rewrite the constants in terms of ω_n, the *natural frequency* of the system and δ, its *damping ratio*

$$\ddot{T}_o + 2\delta\omega_n\dot{T}_o + \omega_n^2 T_o = \left(\dot{Q}R_o + T_e\right)\omega_n^2. \quad (4)$$

The system is critically damped if $\delta = 1$, underdamped if $\delta < 1$ and overdamped when $\delta > 1$,

$$\omega_n^2 = \frac{1}{C_h R_{ho} C_o R_o} \qquad \text{and} \qquad \delta = \frac{\omega_n}{2}\left[C_o R_o + C_h R_o + C_h R_{ho}\right]. \quad (5,6)$$

Proportional Control

An obvious improvement to the on–off control strategy would be to reduce $\dot{Q}$ progressively as the oven temperature approaches the set-point. A proportional controller does this with the function

$$\dot{Q} = \frac{\lambda}{R_o}\left(T_s - T_o\right) \quad (7)$$

subject to the limits $0 \le \dot{Q} \le \dot{Q}_{max}$ (the heater cannot absorb heat). The dimensionless parameter $\lambda > 0$ is the controller *gain*. With this controller the equation 4 becomes

$$\ddot{T}_o + 2\delta_p\omega_p\dot{T}_o + \omega_p^2 T_o = \left(\lambda T_s + T_e\right)\omega_n^2 \quad (8)$$

where

$$\delta_p = \delta \big/ \sqrt{(1+\lambda)} \qquad \text{and} \qquad \omega_p^2 = \omega_n^2(1+\lambda). \quad (9,10)$$

So, increasing λ speeds-up the system but reduces the damping ratio thereby causing overshoot,

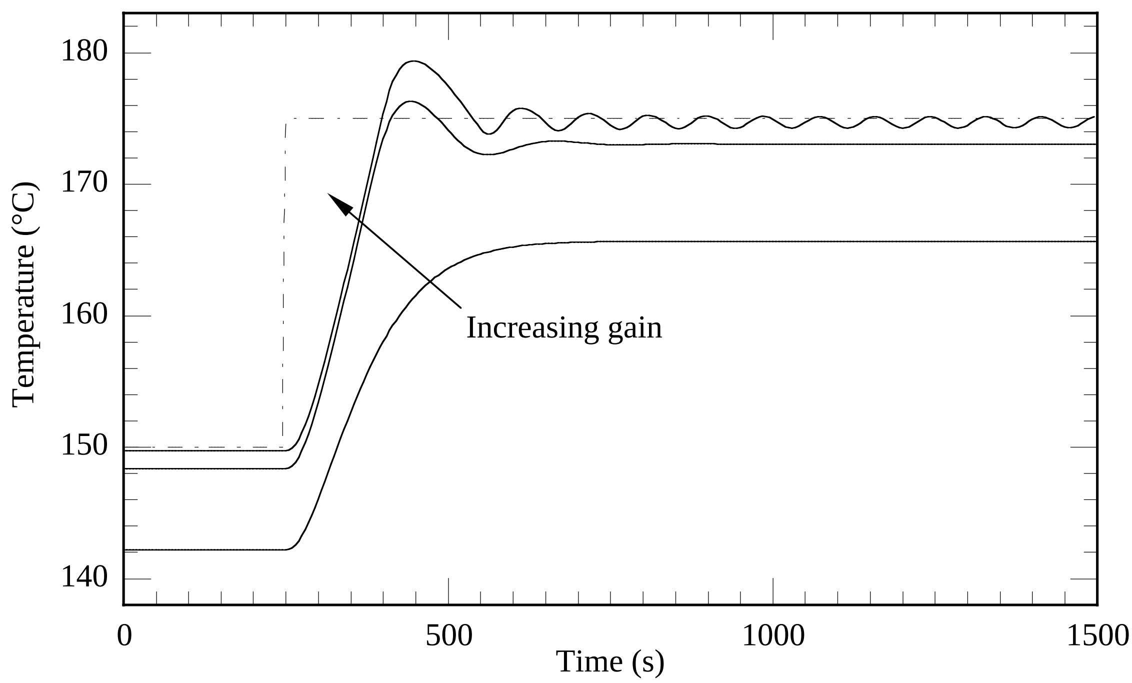

Figure 3. Proportional control. (The power is limited to $\dot{Q}_{max} = 4\,\text{kW}$ during the early part of the transient.)

figure 3, and eventually instability.

The equilibrium oven temperature with this type of controller is not equal to the set-point because the heater power must be zero when $T_o = T_s$. To quantify the the error ΔT set $\dot{T}_o = 0$ and $\ddot{T}_o = 0$ in equation 8 to find

$$\Delta T = (T_s - T_o) = \frac{T_s - T_e}{1 + \lambda}. \tag{11}$$

Derivative Control

The stability problems caused by using high values of gain for proportional control can be mitigated by adding a derivative term to the controller function, *i.e.*

$$\dot{Q} = \frac{\lambda}{R_o}(T_s - T_o) + \frac{\mu}{\omega_n R_o}(\dot{T}_s - \dot{T}_o) \tag{12}$$

where the dimensionless parameter $\mu > 0$ is the controller *damping*. The system equation is now

$$\ddot{T}_o + 2\left(\frac{\delta + \mu/2}{\sqrt{1+\lambda}}\right)\omega_p \dot{T}_o + \omega_p^2 T_o = (\lambda T_s + T_e)\omega_n^2 + \mu\omega_n \dot{T}_s \tag{13}$$

and for any λ a value of μ can be found to satisfy $(\delta + \mu/2) = (1+\lambda)^{1/2}$, the condition for critical damping.

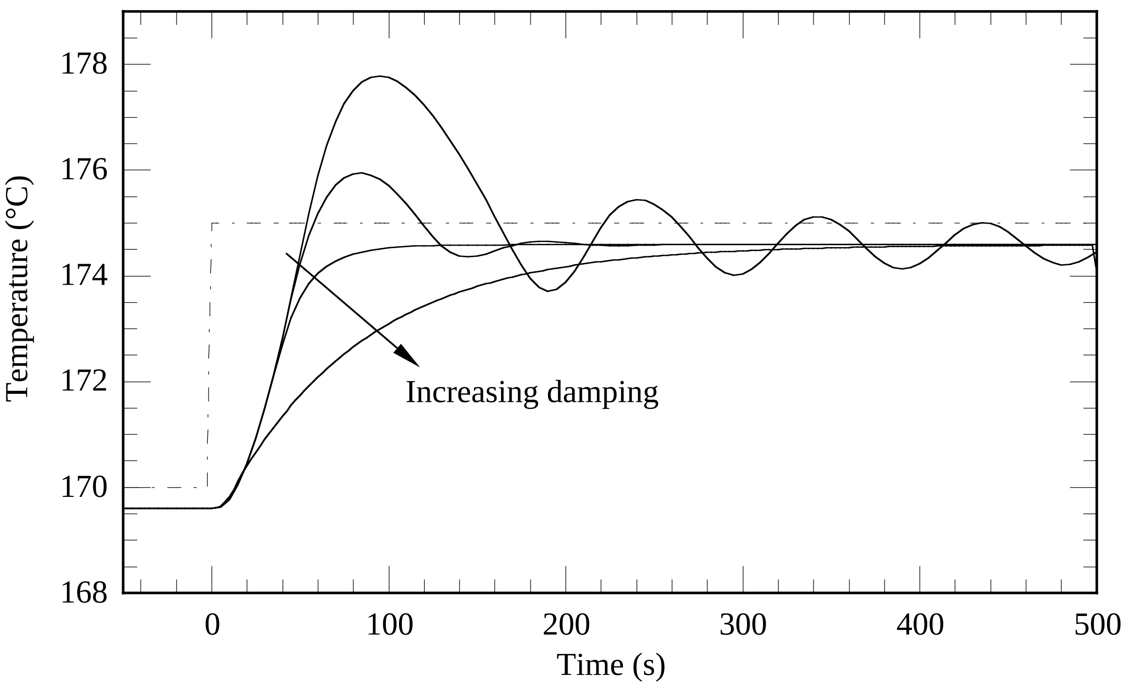

Figure 4. Proportional + derivative control. The proportional gain is kept constant as the damping μ is increased.

Figure 4 shows how adding derivative control improves the performance of a proportional controller. At first sight it would seem that a system with a large λ, and μ adjusted for critical damping, would be a good controller, but in practice there is noise associated with measuring T_o and this severely limits the maximum usable values of λ and μ in a practical system.

Integral Control

Fortunately it is possible to eliminate the steady-state error ΔT while using relatively low gain; this is done by adding an integral term to the control function which becomes

$$\begin{aligned}\dot{Q} &= \frac{\lambda}{R_o}\left(T_s - T_o\right) + \frac{\mu}{\omega_n R_o}\left(\dot{T}_s - \dot{T}_o\right) + \frac{\eta\omega_n}{R_o}\int\left(T_s - T_o\right)\mathrm{d}t \\ &= P \times \left(\left(T_s - T_o\right) + D \times \left(\dot{T}_s - \dot{T}_o\right) + I \times \int\left(T_s - T_o\right)\mathrm{d}t\right)\end{aligned} \tag{14}$$

where the integral level I is often known as the controller *reset* level. This form of function is known as proportional-integral-differential, or PID, control. The effect of the integral term is to change the heater power until the time-averaged value of ΔT is zero. The method works quite well but complicates the analysis slightly because the system is now third-order.

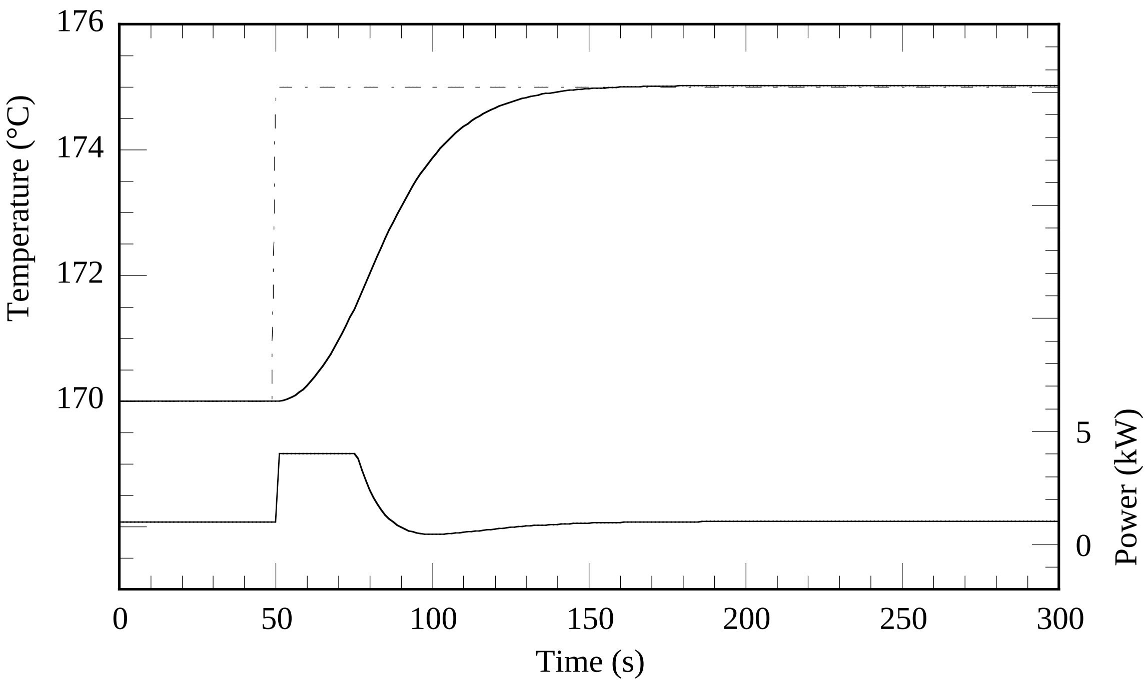

Figure 5. PID control. Adding the integral term has eliminated the steady state error. The slight undershoot in the power suggests that there is scope for further tweaking.

Unlike second-order systems, third-order systems are fairly uncommon in physics but the methods of control theory make the analysis quite straightforward. For instance, applying the so-called

Routh-Hurwitz stability criterion, which is a systematic way of classifying the complex roots of the auxiliary equation, it can be shown that when

$$1 + \lambda > \frac{\eta}{2\delta + \mu}, \tag{15}$$

parameter values can be found to give an acceptably damped response with ΔT eventually tending to zero if the set-point is changed by a step or linear ramp. Whereas derivative control improved the system damping, integral control eliminates steady-state error at the expense of stability margin. In its raw form integral control can be a mixed blessing; if $|T_o - T_s|$ is large for a long period, for example after a large change in T_s or at switch-on, the value of the integral can become excessively large and cause overshoot or undershoot that takes a long time to recover. A sophisticated controller would inhibit integral action until the system is fairly close to equilibrium.

Frequency Response and Noise

The dominant source of noise in most temperature controller systems is equivalent to a white source of spectral density τ_n added to the thermometer signal at the input to the controller. The output of the controller, represented in the frequency domain, is

$$\dot{Q}(\omega) = P\left(1 + Dj\omega - j\,I/\omega\right)\left(\tau_s - \tau_o + \tau_n\right). \tag{16}$$

where τ_o is the Fourier transform of the oven temperature, P, I and D are the controller gain parameters, $j = \sqrt{-1}$, and since the Fourier transform of equation 4

$$\tau_o\left(-\omega^2 + 2j\delta\omega_n\omega + \omega_n^2\right) = \left(\dot{Q}(\omega)R_o + T_e\delta(\omega)\right)\omega_n^2 \tag{17}$$

where $\delta(\omega)$ is the Dirac delta-function, the behaviour of the the system is given by

$$\tau_o = \frac{\left[R_oP(1 + jD\omega - j\,I/\omega)\left(\tau_s + \tau_n\right) + T_e\delta(\omega)\right]\omega_n^2}{\left[\omega_n^2\left(1 + R_oP\right) - \omega^2 + j\omega\omega_n\left(2\delta + \omega_nR_oPD\right) - j\omega_n^2R_oP\,I/\omega\right]}. \tag{18}$$

This has the expected behaviour (see problems A) and shows that as the proportional gain, and hence speed of response, is increased so the noise-induced fluctuations in the oven temperature increase. This means that a compromise between response time and temperature stability has to be made when setting-up such a system.

Practical Systems

Some features of real systems are worth mentioning. Any system using a resistive electrical heater to control temperature is inherently non-linear because an electrical heater can only generate, not absorb, heat. When $T_o > T_s$ cooling occurs at a rate that depends on the oven and its temperature not the controller and *dual PID* controllers allow different heating and cooling parameter values to cope with this. It is possible to build a simple PID controller from a few operational-amplifiers, figure 6. Although the control output should be connected to a 'linear voltage-to-power' converter, the design will work reasonably over a narrow range of temperatures if the output is simply amplified and connected directly to the heater. Commercial PID process controllers vary in cost between £75 for a simple model and £600 for an intelligent autotuning dual PID model.

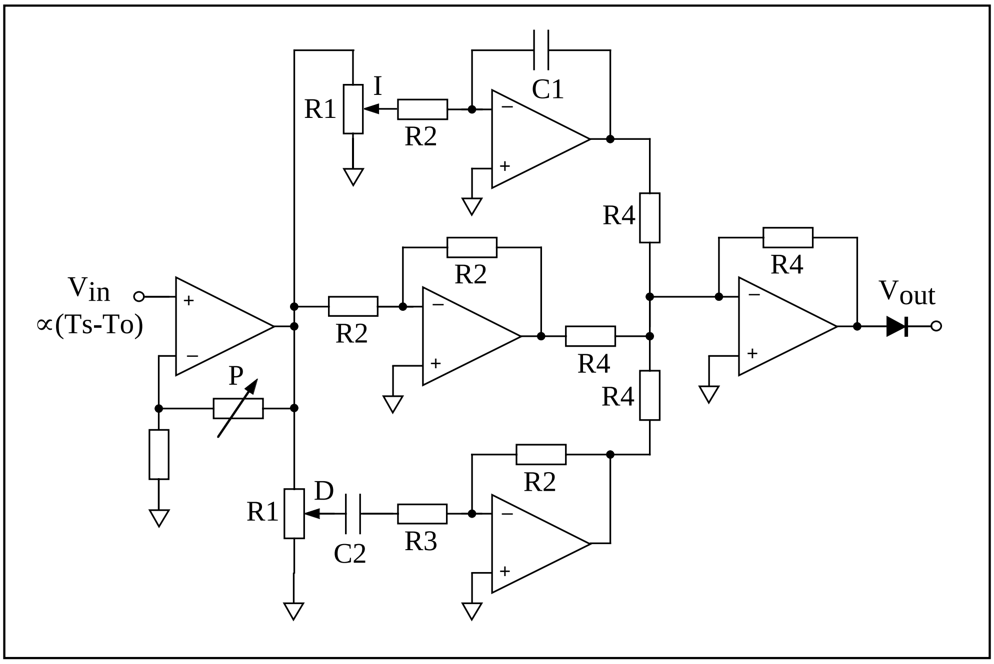

Figure 6. Circuit for home-brew PID control.

Don't assume, without checking, that the knobs on a PID controller correspond to the parameters *P*, *I* and *D* defined in this document. Values are often specified by time constants in which case a long integral time constant is equivalent to a low value of *I* but a long derivative time constant means a large value of *D*. The proportional gain is sometimes set by choosing a 'proportional band' which is the fractional change in temperature that gives maxumum change in heater power so a small number for this corresponds to a large value of *P*. A practical procedure to set up an uncalibrated PID controller, based on the analysis in the previous section, is as follows: (1) Set a typical set-point value T_s, turn off the derivative and integral actions and increase *P* to maximum, or until the system just oscillates; (2) If the system is not oscillating jump to step 4 otherwise reduce *P* by a factor of about two; (3) Observe the effect of increasing *D* on oscillations caused by suddenly decreasing or increasing T_s by about 5% – choose a value for *D* that gives a critically damped response; (4) Slowly increase the integral setting *I* until oscillation just starts, then reduce this value of *I* by a factor of two or three – this should be enough to stop the oscillation. This method is obviously only suitable in cases when the oscillations are not going to be harmful.

Control Theory

Avoid re-inventing the wheel when tackling difficult feedback or control problems – *control theory* is a well developed branch of engineering and has a range of powerful techniques to design and analyse systems involving feedback. As well as having systematic methods for solving complicated problems it introduces the important ideas of *controlability* ('Is it possible to control X by adjusting Y?') and *robustness* ('Will control be regained satisfactorily after an unexpected disturbance?').

Problems

A. Show that the DC (*i.e.* $\omega=0$) behaviour of equation 18 really is as would be expected by examining the cases when (i) $P=0$, (ii) $P\neq0$, $I\neq0$, (iii) $P\neq0$, $I=0$.

B Rewrite equation 15 in terms of P, I and D and hence explain how the constraint affects the terms in equation 18.

C Design a system to control the temperature of a copper disc, thickness 10 mm radius 20 mm. The heater should have a maximum power of 10 W, and control in the range 30–80 °C is required.

There is an interactive version of this document at:

<http://newton.ex.ac.uk/teaching/CDHW/Feedback/index.html>

which includes a simulation that allows the reader to experiment with the performance of the various types of controller.